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ON $*$-SEMIDERIVATIONS AND COMMUTATIVITY OF PRIME $*$-RINGS

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Abstract. In this paper, we introduce the notion of a $*$-semiderivation on $*$-rings, and we try to extend some results for derivations of rings or near-rings to a more general case for $*$-semiderivations of prime $*$-rings.

1. Introduction

Over the last few decades, several authors have investigated the relationship between the commutativity of the ring R and certain specific types of derivations of R. The first result in this direction is due to E. C. Posner [9] who proved that if a ring R admits a nonzero derivation d such that $[d(x), x] \in Z(R)$ for all $x \in R$, then R is commutative. This result was subsequently refined and extended by a number of authors. In [6], Bresar and Vuckman showed that a prime ring must be commutative if it admits a nonzero left derivation. Recently, many authors have obtained commutativity theorems for prime and semiprime rings admitting derivation and generalized derivation. Furthermore, Bresar and Vukman [5] studied the notions of a $*$-derivation and a Jordan $*$-derivation of R. In this paper, we introduce the notion of a $*$-semiderivation on $*$-rings, and we try to extend some results for derivations of rings or near-rings to a more general case for $*$-semiderivations of prime $*$-rings.

2. Preliminaries

Let R be a ring. Then R is *prime* if $aRb = \{0\}$ implies $a = 0$ or $b = 0$. An additive mapping $d : R \to R$ is called a *derivation* if

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$d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping $x \to x^*$ of R into itself is called an *involution* if the following conditions are satisfied;

$$\text{(i) } (xy)^* = y^*x^* \quad \text{(ii) } (x^*)^* = x \text{ for all } x, y \in R.$$

A ring equipped with an involution is called an $*$*-ring* or *ring with involution*. Let R be a $*$-ring. An additive mapping $d : R \to R$ is called an $*$-*derivation* if $d(xy) = d(x)y^* + xd(y)$ holds for all $x, y \in R$.

Definition 2.1. Let R be a prime $*$-ring. An additive mapping $d : R \to R$ is called a $*$-*semiderivation* associated with a surjective function $g : R \to R$ if

(i) $d(xy) = d(x)y^* + g(x)d(y) = d(x)g(y) + x^*d(y)$,
(ii) $d(g(x)) = g(d(x))$ for all $x, y \in R$.

Definition 2.2. Let R be a prime $*$-ring. An additive mapping $d : R \to R$ is called a *reverse* $*$-*semiderivation* associated with a surjective function $g : R \to R$ if

(i) $d(xy) = d(y)x^* + g(y)d(x) = d(y)g(x) + y^*d(x)$,
(ii) $d(g(x)) = g(d(x))$ for all $x, y \in R$.

3. $*$-semiderivations and commutativity of prime $*$-rings

Lemma 3.1. *Let R be a prime $*$-ring and let d be a nonzero $*$-semiderivation associated with g and $a \in R$. If $ad(R) = 0$, then $a = 0$.*

Proof. By hypothesis, we have

$$ad(xy) = 0 \text{ for all } x, y \in R, \tag{3.1}$$

which implies that $ad(x)y^* + ag(x)d(y) = 0$ for all $x, y \in R$. By the hypothesis, we have $ag(x)d(y) = 0$ for all $x, y \in R$. Since g is onto, we get $axd(y) = 0$ for all $x, y \in R$, which implies that $aRd(y) = 0$ for all $y \in R$. Since R is prime and $d \neq 0$, we have $a = 0$. □

Theorem 3.2. *Let R be a prime $*$-ring. If R admits an $*$-semiderivation d associated with g such that $d([x, y]) = 0$ for all $x, y \in R$, then $d = 0$ or R is commutative.*

Proof. By hypothesis, we have

$$d([x, y]) = 0 \text{ for all } x, y \in R. \tag{3.2}$$

Replacing y by yx in (2), we have $d([x, yx]) = d([x, y]x) = d([x, y])x^* + g([x, y])d(x) = 0$ for all $x, y \in R$. By the hypothesis, we get $g([x, y])d(x) =$

0 for all $x, y \in R$. Since g is onto, we have $[x, y]d(x) = 0$ for all $x, y \in R$. Taking zy instead of y with $z \in R$ in this relation, we obtain $[x, z]yd(x) = 0$ for all $x, y, z \in R$. This implies that $[x, z]Rd(x) = \{0\}$ for all $x, z \in R$. Since R is prime, we have $[x, z] = 0$ or $d(x) = 0$ for all $x, z \in R$. Let $K = \{x \in R | d(x) = 0\}$ and $L = \{x \in R | [x, z] = 0, \forall z \in R\}$. Then K and L are both additive subgroups and $K \cup L = R$, but $(R, +)$ is not union of two its proper subgroups, which implies that either $K = R$ or $L = R$. In the former case, we have $d(x) = 0$ for all $x \in R$, that is, $d = 0$. If $L = R$, then we get $[x, z] = 0$ for all $x, y \in R$, which implies that R is commutative. □

THEOREM 3.3. *Let R be a prime $*$-ring. If R admits an $*$-semiderivation d associated with g such that $d(x \circ y) = 0$ for all $x, y \in R$, then $d = 0$ or R is commutative.*

Proof. By hypothesis, we have

$$d(x \circ y) = 0 \text{ for all } x, y \in R. \tag{3.3}$$

Replacing y by yx in (3), we have $d(x \circ yx) = d((x \circ y)x) = d(x \circ y)x^* + g(x \circ y)d(x) = 0$ for all $x, y \in R$. By the hypothesis, we get $g(x \circ y)d(x) = 0$ for all $x, y \in R$. Since g is onto, we have $(x \circ y)d(x) = 0$ for all $x, y \in R$. Taking yx instead of y in this relation, we obtain $(x \circ y)xd(x) = 0$ for all $x, y \in R$. This implies that $(x \circ y)Rd(x) = \{0\}$ for all $x, y \in R$. Since R is prime, we have $x \circ y = 0$ or $d(x) = 0$ for all $x, y \in R$. Let $K = \{x \in R | d(x) = 0\}$ and $L = \{x \in R | x \circ y = 0, \forall y \in R\}$. Then K and L are both additive subgroups and $K \cup L = R$, but $(R, +)$ is not union of two its proper subgroups, which implies that either $K = R$ or $L = R$. In the former case, we have $d(x) = 0$ for all $x \in R$, that is, $d = 0$. If $L = M$, then we get $x \circ y = 0$ for all $x, y \in R$, which implies that $xy = -yx$ for all $x, y \in R$. Again, replacing x by xz in the last relation, we have $xzy = -yxz = xyz$, that is, $x[z, y] = 0$ for all $x, y, z \in R$. This implies that $R[z, y] = \{0\}$ for all $x, z \in R$. Hence $tR[z, y] = \{0\}$ for all $0 \neq t, y, z \in R$. Since R is prime, we have $[z, y] = 0$ for all $y, z \in R$, which implies that R is commutative. □

THEOREM 3.4. *Let R be a prime $*$-ring. If R admits an $*$-semiderivation d associated with g such that $[d(x), y] = 0$ for all $x, y \in R$, then $d = 0$ or R is commutative.*

Proof. By hypothesis, we have

$$[d(x), y] = 0 \text{ for all } x, y \in R. \tag{3.4}$$

Replacing x by xz in (4) and using (4), we have

$$\begin{aligned}
0 &= [d(xz), y] = [d(x)z^* + g(x)d(z), y] \\
&= [d(x)z^*, y] + [g(x)d(z), y] \\
&= d(x)[z^*, y] + [d(x), y]z^* + g(x)[d(z), y] + [g(x), y]d(z) \\
&= d(x)[z^*, y] + [g(x), y]d(z)
\end{aligned} \tag{3.5}$$

for all $x, y, z \in R$. Taking $g(x)$ instead of y in (5), we have $d(x)[z^*, g(x)] = 0$ for all $x, z \in R$. Substituting z^* for z in this relation, we get $d(x)[z, g(x)] = 0$ for all $x, z \in R$. Again, replacing z by zy in the last relation, we obtain $d(x)z[y, g(x)] = 0$ for all $x, y, z \in R$. Hence $d(x)R[y, g(x)] = 0$ for all $x, y \in R$. Since R is prime, we have $d(x) = 0$ or $[y, g(x)] = 0$ for all $x, y \in R$. Let

$$K = \{x \in R | d(x) = 0\} \text{ and } L = \{x \in R | [y, g(x)] = 0, \forall y \in R\}.$$

Then K and L are both additive subgroups and $K \cup L = R$, but $(R, +)$ is not union of two its proper subgroups, which implies that either $K = R$ or $L = R$. In the former case, we have $d(x) = 0$ for all $x \in R$, that is, $d = 0$. If $L = R$, then we get $[y, g(x)] = 0$ for all $x, y \in R$. Since g is onto, we have $[y, x] = 0$ for all $x, y \in R$, which implies that R is commutative. □

THEOREM 3.5. *Let R be a prime $*$-ring. If R admits an $*$-semiderivation d associated with g such that $d(x) \circ y = 0$ for all $x, y \in R$, then $d = 0$ or R is commutative.*

Proof. By hypothesis, we have

$$d(x) \circ y = 0 \text{ for all } x, y \in R. \tag{3.6}$$

Replacing x by xz in (6) and using (6), we have

$$\begin{aligned}
0 &= d(xz) \circ y = (d(x)z^* + y + g(x)d(z)) \circ y \\
&= d(x)z^* \circ y + g(x)d(z) \circ y \\
&= (d(x) \circ y)z^* + d(x)[z^*, y] + g(x)(d(z) \circ y) - [g(x), y]d(z) \\
&= d(x)[z^*, y] - [g(x), y]d(z)
\end{aligned} \tag{3.7}$$

for all $x, y, z \in R$. Taking $g(x)$ instead of y in (7), we have $d(x)[z^*, g(x)] = 0$ for all $x, z \in R$. Substituting z^* for y in this relation, we get $d(x)[z, g(x)] = 0$ for all $x, z \in R$. Again, replacing z by zy in the last relation, we obtain $d(x)z[y, g(x)] = 0$ for all $x, y, z \in R$. Hence $d(x)R[y, g(x)] = 0$ for all $x, y \in R$. Since R is prime, we have $d(x) = 0$ or $[y, g(x)] = 0$ for all $x, y \in R$. Let $K = \{x \in R | d(x) = 0\}$ and $L = \{x \in R | [y, g(x)] = 0, \forall y \in R\}$. Then K and L are both additive subgroups and $K \cup L = R$, but $(R,+)$ is not union of two its proper subgroups, which implies that either $K = R$ or $L = R$. In the former case, we have $d(x) = 0$ for all $x \in R$, that is, $d = 0$. If $L = R$, then we get $[y, g(x)] = 0$ for all $x, y \in R$. Since g is onto, we have $[y, x] = 0$ for all $x, y \in R$, which implies that R is commutative. □

THEOREM 3.6. *Let R be a prime $*$-ring and let d be an $*$-semiderivation associated with g such that g is an automorphism of R. If $d(xy) = d(x)d(y)$ for all $x, y \in R$, then $d = 0$.*

Proof. For any $x, y \in R$, we have

$$d(xy) = d(x)y^* + g(x)d(y) = d(x)d(y) \text{ for all } x, y \in R. \tag{3.8}$$

Replacing x by xw in (8), we obtain $d(xw)y^* + g(xw)d(y) = d(xw)d(y)$ for all $x, y, w \in R$. Hence $d(x)d(w)y^* + g(x)g(w)d(y) = d(x)d(w)d(y) = d(x)d(wy)$ for all $x, y, w \in R$, and hence $d(x)d(w)y^* + g(x)g(w)d(y) = d(x)d(w)y^* + d(x)g(w)d(y)$ for all $x, y, w \in R$. This implies that $(g(x) - d(x))g(w)d(y) = 0$ for all $x, y, w \in R$. Since R is prime and g is an automorphism of R, we have $d(x) = g(x)$ or $d(y) = 0$ for all $x, y \in R$. Let us assume that $d(x) = g(x)$ for all $x \in R$. Substituting xy for x in the last relation, we have $d(x)y^* + g(x)d(y) = g(x)g(y) = g(x)d(y)$ for all $x, y \in R$, that is, $d(x)g(y^*) = 0$ for all $x, y \in R$. Taking y^* instead of y in this relation, we have $d(x)g(y) = 0$ for all $x, y \in R$. Again, replacing y by $g^{-1}(y)$ in the last relation, we have $d(x)y = 0$, which implies that $d(x)R = \{0\}$ for all $x \in R$. Thus we obtain $d(x) = 0$ for all $x \in R$ in any case. □

THEOREM 3.7. *Let R be a prime $*$-ring and let d be an *-semiderivation associated with g. If $d(xy) = d(y)d(x)$ for all $x, y \in R$ and $d(x) \neq x^*$ for all $x \in R$, then $d = 0$.*

Proof. For any $x, y \in R$, we have

$$d(xy) = d(x)y^* + g(x)d(y) = d(y)d(x), \text{ for all } x, y \in R. \tag{3.9}$$

Replacing y by xy in (9), we obtain $d(x)(xy)^* + g(x)d(xy) = d(xy)d(x)$ for all $x, y \in R$. Hence we have

$$d(x)y^*x^* + g(x)d(y)d(x) = d(x)y^*d(x) + g(x)d(y)d(x)$$

for all $x, y \in R$, and hence $d(x)y^*x^* = d(x)y^*d(x)$ for all $x, y \in R$. This implies that $d(x)y^*(x^* - d(x)) = 0$ for all $x, y \in R$. Substituting y^* for y in the last relation, we get $d(x)y(x^* - d(x)) = 0$ for all $x, y \in R$. That is, $d(x)R(x^* - d(x)) = \{0\}$ for all $x \in R$. Since R is prime, we have $d(x) = x^*$ or $d(x) = 0$ for all $x \in R$. But $d(x) \neq x^*$ for all $x \in R$, and so $d(x) = 0$ for all $x \in R$. □

THEOREM 3.8. *Let R be a prime $*$-ring and let d be an $*$-semiderivation associated with g such that $g(xy) = g(x)g(y)$ for all $x, y \in R$. Then $d = 0$ or R is commutative.*

Proof. By hypothesis, we have

$$d(xy) = d(x)y^* + g(x)d(y), \text{ for all } x, y \in R. \tag{3.10}$$

Replacing y by yz in (10), we have $d(xyz) = d(x)(yz)^* + g(x)d(yz)$ for all $x, y, z \in R$. Hence we get

$$\begin{aligned} d(xyz) &= d(x)z^*y^* + g(x)(d(y)z^* + g(y)d(z)) \\ &= d(x)z^*y^* + g(x)d(y)z^* + g(x)g(y)d(z) \end{aligned} \tag{3.11}$$

for all $x, y, z \in R$. On the other hand, we get

$$\begin{aligned} d(xyz) &= d(xy(z)) \\ &= d(xy)z^* + g(xy)d(z) \\ &= d(x)y^*z^* + g(x)d(y)z^* + g(x)g(y)d(z) \end{aligned} \tag{3.12}$$

for all $x, y, z \in R$. Comparing (11) and (12), we have $d(x)[z^*, y^*] = 0$ for all $x, y \in R$. Replacing z by z^* and y by y^* in this relation, we obtain

$$d(x)[z, y] = 0 \text{ for all } x, y, z \in R. \tag{3.13}$$

Substituting y by yt with $t \in R$ in (13), we have

$$\begin{aligned} 0 = d(x)[z, yt] &= d(x)[z, y]t + d(x)y[z, t] \\ &= d(x)y[z, t] \end{aligned}$$

for every $t, x, y, z \in R$. Hence $d(x)R[z, t] = \{0\}$ for every $t, x, z \in R$. Since R is prime, we have $d(x) = 0$ or $[z, t] = 0$ for all $t, x, z \in R$. Let $K = \{x \in R | d(x) = 0\}$ and $L = \{z \in R | [z, t] = 0, \forall t \in R\}$. Then K and L are both additive subgroups and $K \cup L = R$, but $(R, +)$ is not union of two its proper subgroups, which implies that either $K = R$ or $L = R$. In the former case, we have $d = 0$. If $L = R$, then $[z, t] = 0$ for all $t, z \in R$, which implies that R is commutative. □

THEOREM 3.9. *Let R be a prime $*$-ring and let d be a reverse $*$-semiderivation associated with g such that $g(xy) = g(x)g(y)$ for all $x, y \in R$. Then $[d(x), z] = 0$ for all $x, z \in R$ or $d = 0$.*

Proof. By hypothesis, we have

$$d(xy) = d(y)x^* + g(y)d(x) \text{ for all } x, y \in R. \tag{3.14}$$

Replacing x by xz in (14), we have

$$\begin{aligned} d(xzy) &= d(y)z^*x^* + g(y)(d(z)x^* + g(z)d(x)) \\ &= d(y)z^*x^* + g(y)d(z)x^* + g(y)g(z)d(x) \end{aligned} \tag{3.15}$$

for all $x, y, z \in R$. On the other hand,

$$\begin{aligned} d(xzy) = d(x(zy)) &= d(zy)x^* + g(zy)d(x) \\ &= d(y)z^*x^* + g(y)d(z)x^* + g(z)g(y)d(x) \\ &= d(y)z^*x^* + g(y)d(z)x^* + g(z)g(y)d(x) \end{aligned} \tag{3.16}$$

Comparing (15) with (16), we get $[g(z), g(y)]d(x)$ for all $x, y, z \in R$. Since g is onto, we obtain $[z, y]d(x)$ for all $x, z \in R$. Again, replacing y by $d(x)z$ in this relation, we have

$$\begin{aligned} 0 &= [d(x)z, z]d(x) \\ &= d(x)[z, z]d(x) + [d(x), z]zd(x) \\ &= [d(x), z]zd(x). \end{aligned} \tag{3.17}$$

Since R is prime, we can get either $[d(x), z] = 0$ or $d(x) = 0$ for all $x, z \in R$. □

Theorem 3.10. *Let R be a prime ∗-ring and let d be an ∗-semiderivation associated with g. If $d(x) \circ g(y) = 0$ for all $x, y \in R$, then $d = 0$ or R is commutative.*

Proof. By hypothesis, we have

$$d(x) \circ g(y) = 0 \text{ for all } x, y \in R. \tag{3.18}$$

Replacing x by yx in (18), we have

$$\begin{aligned} 0 &= d(yx) \circ g(y) \\ &= (d(y)x^* + g(y)d(x)) \circ g(y) \\ &= d(y)x^* \circ g(y) + g(y)d(x) \circ g(y) \\ &= (d(y) \circ g(y))x^* + d(y)[x^*, g(y)] + g(y)(d(x) \circ g(y)) \\ &\quad - [g(y), g(y)]d(x) \\ &= d(y)[x^*, g(y)] \end{aligned} \tag{3.19}$$

for every $x, y \in R$. Substituting x^* for x in (19), we get $d(y)[x, g(y)] = 0$ for all $x, y \in R$. Taking xz instead of x in this relation, we obtain $d(y)x[z, g(y)] = 0$ for all $x, y, z \in R$. This implies that $d(y)R[z, g(y)] = \{0\}$ for all $y, z \in R$. Since R is prime, we have $d(y) = 0$ or $[z, g(y)] = 0$ for all $y, z \in R$. Let $K = \{y \in R | d(y) = 0\}$ and $L = \{y \in R | [z, g(y)] = 0, \forall z \in R\}$. Then K and L are both additive subgroups and $K \cup L = R$, but $(R,+)$ is not union of two its proper subgroups, which implies that either $K=R$ or $L=R$. In the former case, we have $d=0$. If $L=R$, then $[z,g(y)]=0$ for all $y,z\in R$. Since g is onto, we get $[z,y]=0$ for all $y,z\in R$, which implies that R is commutative. $\square$

THEOREM 3.11. *Let R be a prime $*$-ring and let d be an $*$-semiderivation associated with g. If $[d(x),g(y)]=0$ for all $x,y\in R$, then $d=0$ or R is commutative.*

Proof. By hypothesis, we have

$$[d(x),g(y)]=0 \text{ for all } x,y\in R. \tag{3.20}$$

Replacing x by yx in (18), we have

$$\begin{aligned} 0 &= [d(yx),g(y)] \\ &= [d(y)x^*+g(y)d(x),g(y)] \\ &= [d(y)x^*,g(y)]+[g(y)d(x),g(y)] \\ &= d(y)[x^*,g(y)]+[d(y),g(y)]x^*+g(y)[d(x),g(y)] \\ &\quad +[g(y),g(y)]d(x) \\ &= d(y)[x^*,g(y)] \end{aligned} \tag{3.21}$$

for every $x,y\in R$. Substituting x^* for x in (21), we get $d(y)[x,g(y)]=0$ for all $x,y\in R$. Taking xz instead of x in this relation, we obtain $d(y)x[z,g(y)]=0$ for all $x,y,z\in R$. This implies that $d(y)R[z,g(y)]=\{0\}$ for all $y,z\in R$. Since R is prime, we have $d(y)=0$ or $[z,g(y)]=0$ for all $y,z\in R$. Let $K=\{y\in R|d(y)=0\}$ and $L=\{y\in R|[z,g(y)]=0,\forall z\in R\}$. Then K and L are both additive subgroups and $K\cup L=R$, but $(R,+)$ is not union of two its proper subgroups, which implies that either $K=R$ or $L=R$. In the former case, we have $d=0$. If $L=R$, then $[z,g(y)]=0$ for all $y,z\in R$. Since g is onto, we get $[z,y]=0$ for all $y,z\in R$, which implies that R is commutative. $\square$

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